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CRACK NUCLEATION DUE TO BENDING OF A VARIABLE-THICKNESS BAND

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A mathematical model of crack nucleation in a band (beam) bent in its plane by a given system of external loads (constant bending moments, uniformly distributed pressure, etc.) is constructed. Pre-fracture zones modeled as zones of attenuated interparticle bonds of the material are assumed to appear in the variable-thickness band in the course of its loading. The solution of the problem of equilibrium of an isotropic band of variable thickness with a nucleating crack is reduced to the solution of a nonlinear singular integrodifferential equation with a kernel of the Cauchy kernel type, which yields the forces in the zone of crack nucleation. The condition of crack nucleation in the variable-thickness band is formulated with allowance for the criterion of critical tension of the material bonds.

Key words: *variable-thickness band, pre-fracture zone, bonds between the faces.*

Formulation of the Problem. Plates (rods) of variable thickness are often used to reduce the mass of thin-walled structures, because articles with required properties can be obtained by varying the thickness of the sheet materials used. Adequate methods of estimating the load-bearing capacity of variable-thickness plates (bands) with cracks have not been yet developed. It is important to study the process of variable-thickness band (beam) fracture to solve problems of practical importance.

Let us consider a uniform isotropic band (beam) of variable thickness. The band width and thickness are denoted by $2c$ and $2h$, respectively (Fig. 1). The mid-surface (x, y) is the plane of symmetry. The variable-thickness band is in a generalized plane stress state. As the band is loaded, pre-fracture zones arise in the material, which are modeled as zones with attenuated interparticle bonds in the material. Interaction of the faces of these zones is modeled by introducing bonds with a specified strain diagram between the faces of the pre-fracture zone. The physical nature of such bonds and the sizes of the pre-fracture zones depend on the type of the material. As these zones (interlayers of a super-stressed material) are small as compared with the remaining elastic part of the band, they can be conceptually removed and replaced by cuts whose surfaces interact with each other in accordance with a certain law corresponding to the action of the removed material. Allowance for these effects in problems of fracture mechanics is an important but a difficult problem.

In the case considered, crack nucleation in the variable-thickness band is the process of transformation of the pre-fracture zone to the zone of broken bonds between the material surfaces. The size of the zone with attenuated interparticle bonds of the material is not known in advance and has to be determined in the course of solving the problem.

Zones with a violated structure of the material are known to be formed at the early stages of fracture. These zones are shaped as narrow layers that occupy an insignificant volume of the body, as compared with its elastic zone [1–3]. The band (beam) thickness $2h(x, y)$ is assumed to satisfy the conditions $0 < h_1 \leqslant h(x, y) \leqslant h_2$, where h_1 and h_2 are the smallest and the greatest values of the band thickness, respectively.

The function of the band thickness can be presented as

$$h(x, y) = h_0[1 + \varepsilon\bar{h}(x, y)],$$

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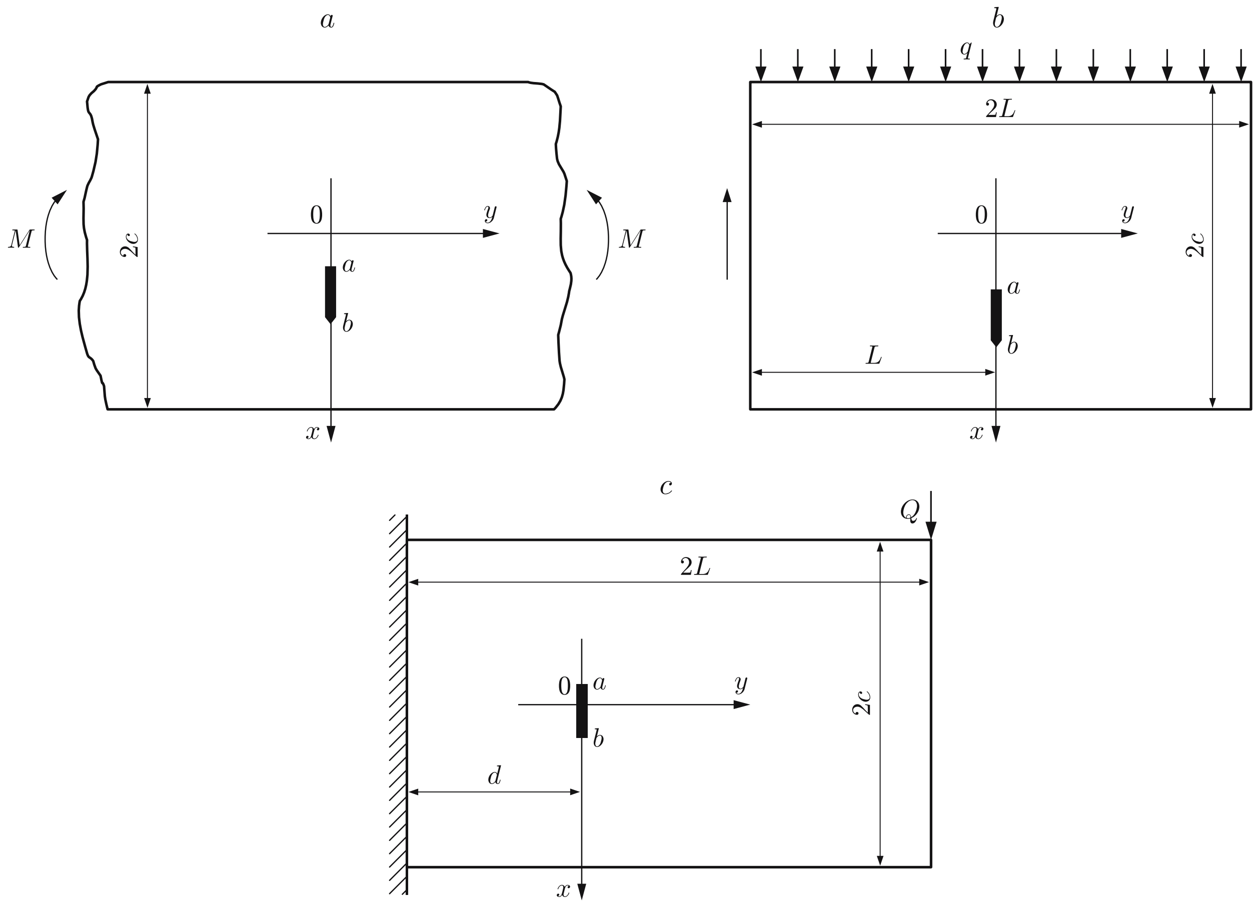

Fig. 1. Scheme of the problem of crack nucleation in a variable-thickness band under bending: (a) pure bending; (b) uniformly distributed load; (c) bending of a beam fixed at one end by a concentrated force Q.

where $h_0 = (h_1 + h_2)/2$, $\varepsilon = (h_2 - h_1)/(h_2 + h_1)$, and $-1 \leqslant \bar{h}(x, y) \leqslant 1$ is a certain known dimensionless continuous function.

Let us use the following assumptions. The band is subjected to external loading (bending moments, pressure uniformly distributed along the band, or concentrated forces) applied in the mid-plane of the band (see Fig. 1). The band faces parallel to the plane xy are free from external loading. The pre-fracture zone is aligned in the direction of the maximum tensile stresses. The x axis of the coordinate system (xy) coincides with the line along which the pre-fracture zone is located ($a \leqslant x \leqslant b$). Interaction of the pre-fracture zone faces (bonds between the faces) prevents crack nucleation.

The mathematical description of interaction of the pre-fracture zone faces implies that there are bonds between these faces, which follow a specified deformation law. The force load on the band induces normal $q_y(x)$ and shear $q_{xy}(x)$ forces in the bonds connecting the pre-fracture zone faces in the general case. Therefore, normal stresses $q_y(x)$ and shear stresses $q_{xy}(x)$ are applied to the faces of the zone of attenuated interparticle bonds of the material. These stresses are not known in advance and have to be determined in the course of solving the boundary-value problem of fracture mechanics for the band.

Let us write the equations of static deformation of the band:
— equilibrium equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \qquad \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0;$$

— Hooke's law

$$N_x = \frac{2Eh}{1-\nu^2}\Big(\frac{\partial u}{\partial x} + \nu\,\frac{\partial v}{\partial y}\Big), \qquad N_y = \frac{2Eh}{1-\nu^2}\Big(\frac{\partial v}{\partial y} + \nu\,\frac{\partial u}{\partial x}\Big), \qquad N_{xy} = \frac{Eh}{1+\nu}\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big).$$

Here, N_x, N_y, and N_{xy} are the normal and shear forces per unit length, respectively, u and v are the components of the displacement vector, E is Young's modulus of the band material, and ν is Poisson's ratio of the material.

The boundary condition in the pre-fracture zone has the form

$$\sigma_y - i\tau_{xy} = q_y - iq_{xy} \quad \text{for} \quad y = 0, \qquad a \leqslant x \leqslant b. \tag{1}$$

The constitutive relations of the problem posed should be supplemented with an equation relating the opening of the pre-fracture zone faces and the forces in the bonds. Without loss of generality, this equation can be presented as [3]

$$v^+(x,0) - v^-(x,0) - i(u^+(x,0) - u^-(x,0)) = C(x,\sigma)[q_y(x) - iq_{xy}(x)], \tag{2}$$

where the function $C(x,\sigma)$ can be considered as the effective compliance of the bonds, depending on their tension, $v^+ - v^-$ and $u^+ - u^-$ are the normal and shear components of the opening of the pre-fracture zone faces, respectively, and $\sigma = \sqrt{q_y^2 + q_{xy}^2}$ is the absolute value of the force vector in the bonds.

Method of the Solution. The solution of the system of equations of static deformation of the band is sought in the form

$$N_x = N_x^{(0)} + \varepsilon N_x^{(1)} + \ldots, \qquad N_y = N_y^{(0)} + \varepsilon N_y^{(1)} + \ldots, \qquad N_{xy} = N_{xy}^{(0)} + \varepsilon N_{xy}^{(1)} + \ldots,$$

$$u = u_0 + \varepsilon u_1 + \ldots, \quad v = v_0 + \varepsilon v_1 + \ldots, \quad a = a_0 + \varepsilon a_1 + \ldots, \quad b = b_0 + \varepsilon b_1 + \ldots, \tag{3}$$

$$q_y = q_y^{(0)} + \varepsilon q_y^{(1)} + \ldots, \qquad q_{xy} = q_{xy}^{(0)} + \varepsilon q_{xy}^{(1)} + \ldots.$$

In constructing the solution, we use procedure (3) of the perturbation method. The equations of the zeroth approximation coincide with the equations of the classical plane problem of the elasticity theory, and the equations of the first approximation are the equations of the plane problem of the elasticity theory with the volume force

$$X_1 = N_x^{(0)}\,\frac{\partial \bar h}{\partial x} + N_{xy}^{(0)}\,\frac{\partial \bar h}{\partial y}, \qquad Y_1 = N_y^{(0)}\,\frac{\partial \bar h}{\partial y} + N_{xy}^{(0)}\,\frac{\partial \bar h}{\partial x}. \tag{4}$$

The components of the volume force for the second and next approximations are determined in a similar manner.

The boundary conditions of problem (1) have the following form:

— in the zeroth approximation,

$$N_y^{(0)} - iN_{xy}^{(0)} = q_y^{*(0)} - iq_{xy}^{*(0)} \quad \text{for} \quad y = 0, \qquad a_0 \leqslant x \leqslant b_0; \tag{5}$$

— in the first approximation,

$$N_y^* - iN_{xy}^* = -\bar h(x,0)(q_y^{*(0)} - iq_{xy}^{*(0)}) + q_y^{*(1)} - iq_{xy}^{*(1)} \quad \text{for} \quad y = 0, \qquad a_1 \leqslant x \leqslant b_1. \tag{6}$$

Here, $q_y^* - iq_{xy}^* = 2h_0(q_y - iq_{xy})$.

The following notation was used in deriving the equations in the first approximation:

$$N_x^* = N_x^{(1)} - N_{x_0}^{(1)}, \qquad N_{x_0}^{(1)} = \bar h(x,y)N_x^{(0)}, \qquad N_y^* = N_y^{(1)} - N_{y_0}^{(1)}, \qquad N_{y_0}^{(1)} = \bar h(x,y)N_y^{(0)},$$

$$N_{xy}^* = N_{xy}^{(1)} - N_{xy_0}^{(1)}, \qquad N_{xy_0}^{(1)} = \bar h(x,y)N_{xy}^{(0)}.$$

Equation (2) yields the following relations:

$$v_0^+ - v_0^- - i(u_0^+ - u_0^-) = C(x,\sigma^0)[q_y^{(0)}(x) - iq_{xy}^{(0)}(x)] \tag{7}$$

in the zero approximation, and

$$v_1^+ - v_1^- - i(u_1^+ - u_1^-) = C(x,\sigma^1)[q_y^{(1)}(x) - iq_{xy}^{(1)}(x)] \tag{8}$$

in the first approximation, Here, $\sigma^j = \sqrt{[q_y^{(j)}]^2 + [q_{xy}^{(j)}]^2}$ $(j = 0, 1)$.

Let us consider the solution in the zeroth approximation. Under the conditions of the plane problem of the elasticity theory, the components $N_x^{(0)}$, $N_y^{(0)}$, and $N_{xy}^{(0)}$ of the stress tensor and the components u_0 and v_0 of the displacement vector are expressed via two analytical functions $\Phi^{(0)}(z)$ and $\Omega^{(0)}(z)$ [4]. The stress–strain state in the vicinity of the pre-fracture zone is determined approximately [4]: the boundary conditions of the problem [conditions (5) and (6)] are satisfied on the contour of the pre-fracture zone, and the stress state in the band at a significant distance from the pre-fracture zone coincides with the stress state determined by the functions

$$\lim_{|z|\to\infty} \Phi^{(0)}(z) = \Phi_0(z) = A_0 z^3 + A_1 z^2 + A_2 z + A_3,$$
$$\lim_{|z|\to\infty} \Omega^{(0)}(z) = \Omega_0(z) = B_0 z^3 + B_1 z^2 + B_2 z + B_3. \tag{9}$$

For different values of the coefficients A_j and B_j ($j = 0, 1, 2, 3$), these functions determine the stress state in the band (beam) in the absence of the pre-fracture zone. For instance, assuming that

$$A_0 = 0, \qquad A_1 = 0, \qquad A_2 = M/(4I), \qquad A_3 = 0,$$
$$B_0 = 0, \qquad B_1 = 0, \qquad B_2 = 3M/(4I), \qquad B_3 = 0 \tag{10}$$

in Eqs. (9) (I is the moment of inertia of the band cross section), we can show that the functions $\Phi_0(z)$ and $\Omega_0(z)$ determine the solution of the problem of pure bending of an infinite band (beam) by moments M in the absence of the pre-fracture zone (see Fig. 1a).

Similarly, for

$$A_0 = q/(24I), \qquad A_1 = 0, \qquad A_2 = q(L^2 + 3c^2/5)/(8I), \qquad A_3 = -qc^3/(12I),$$
$$B_0 = 7q/(24I), \qquad B_1 = 0, \qquad B_2 = q(3L^2 - 11c^2/5)/(8I), \qquad B_3 = qc^3/(12I), \tag{11}$$

the functions $\Phi_0(z)$ and $\Omega_0(z)$ yield the solution of the problem of bending of a beam of length $2L$ loaded by uniform pressure of intensity q in the absence of the pre-fracture zone (see Fig. 1b). In this case, the beam is assumed to be freely located on two supports, and the support responses are determined as the shear forces applied to the end faces of the beam. For

$$A_0 = 0, \qquad A_1 = -iQ/(8I), \qquad A_2 = -Q(2L - d)/(4I), \qquad A_3 = 0,$$
$$B_0 = 0, \qquad B_1 = 5iQ/(8I), \qquad B_2 = -3Q(2L - d)/(4I), \qquad B_3 = -iQc^2/(2I), \tag{12}$$

the functions $\Phi_0(z)$ and $\Psi_0(z)$ yield the solution of the problem of bending of a rigidly fixed cantilever beam in the absence of the pre-fracture zone under the action of a constant transverse force Q applied to the free end of the beam (see Fig. 1c).

The presence of the pre-fracture zone in the band disturbs the field of elastic stresses in its vicinity. The stress–strain state in the band far from the pre-fracture zone under the above-mentioned loads is determined by Eqs. (9) if the values of the coefficients A_j and B_j are determined by equalities (10)–(12).

The boundary-value problem (5) is reduced to the problem of linear coupling of the boundary conditions of the functions $\Phi^{(0)}(z)$ and $\Omega^{(0)}(z)$ [4]

$$[\Phi^{(0)}(t) + \Omega^{(0)}(t)]^+ + [\Phi^{(0)}(t) + \Omega^{(0)}(t)]^- = 2f(t),$$
$$[\Phi^{(0)}(t) - \Omega^{(0)}(t)]^+ - [\Phi^{(0)}(t) - \Omega^{(0)}(t)]^- = 0, \tag{13}$$

where $a_0 \leqslant t \leqslant b_0$, t is the affix of the points of the pre-fracture zone contour, and $f(t) = q_y^{(0)} - iq_{xy}^{(0)}$.

Solving problem (13) in the class of everywhere bounded functions, we find

$$\Phi^{(0)}(z) = \frac{\sqrt{(z-a_0)(z-b_0)}}{2\pi i} \int_{a_0}^{b_0} \frac{f(t)\,dt}{\sqrt{(t-a_0)(t-b_0)}\,(t-z)} + \sqrt{(z-a_0)(z-b_0)}\,P_n(z) + \frac{1}{2}\,[\Phi_0(z) - \Omega_0(z)],$$
$$\Omega^{(0)}(z) = \frac{\sqrt{(z-a_0)(z-b_0)}}{2\pi i} \int_{a_0}^{b_0} \frac{f(t)\,dt}{\sqrt{(t-a_0)(t-b_0)}\,(t-z)} + \sqrt{(z-a_0)(z-b_0)}\,P_n(z) - \frac{1}{2}\,[\Phi_0(z) - \Omega_0(z)], \tag{14}$$

where the functions $\Phi_0(z)$ and $\Omega_0(z)$ are determined by equalities (9); the polynomial $P_n(z)$ has the form

$$P_n(z) = D_n z^n + D_{n-1} z^{n-1} + \ldots + D_0. \tag{15}$$

The power of polynomial (15) and its coefficients $D_0, D_1, \ldots, D_n$ are determined by the behavior of the functions $\Phi^{(0)}(z)$ and $\Omega^{(0)}(z)$ in the neighborhood $|z| = \infty$. The functions $\Phi^{(0)}(z)$ and $\Omega^{(0)}(z)$ are analytical in the domain outside the pre-fracture zone and have the following form at large values of $|z|$:

$$\Phi^{(0)}(z) = \Phi_0(z) + O(1/z^2), \qquad \Omega^{(0)}(z) = \Omega_0(z) + O(1/z^2). \tag{16}$$

To determine the coefficients $D_0, D_1, \ldots, D_n$ and the values of a_0 and b_0, it is necessary to expand the function $\Phi^{(0)}(z)$ into a series with respect to the powers of z in the neighborhood of the point $|z| = \infty$ and to compare this expansion with Eq. (16). Taking into account the above-given relations and performing necessary calculations, we obtain the system of equations

$$\begin{gathered} D_2 + (A_0 - B_0)/2 = A_0, \qquad D_1 - (a_0 + b_0) D_2/2 + (A_1 - B_1)/2 = A_1, \\ D_0 - (a_0 + b_0) D_1/2 - (a_0 - b_0)^2 D_2/8 + (A_2 - B_2)/2 = A_2, \\ -C_1 - (a_0 + b_0) D_0/2 - (a_0 - b_0)^2 D_1/2 + (A_3 - B_3)/2 = A_3, \\ (a_0 + b_0) C_1/2 - C_2 - (a_0 - b_0)^2 D_0/8 = 0, \qquad D_n = 0, \qquad n \geqslant 3, \end{gathered} \tag{17}$$

where

$$C_1 = \frac{1}{2\pi i} \int\limits_{a_0}^{b_0} \frac{f(t)\, dt}{\sqrt{(t - a_0)(t - b_0)}}, \qquad C_2 = \frac{1}{2\pi i} \int\limits_{a_0}^{b_0} \frac{t f(t)\, dt}{\sqrt{(t - a_0)(t - b_0)}}.$$

The last two equations in system (17) allow us to determine the parameters a_0 and b_0.

Formulas (14) and Eqs. (17) involve unknown stresses in the pre-fracture zone. The condition determining the unknown stresses in the bonds between the pre-fracture zone faces in the zeroth approximation is the additional relation (7). Using solution (14), we find the opening of the opposite faces of the pre-fracture zone:

$$2i\mu \frac{\partial}{\partial x} [v_0^+(x,0) - v_0^-(x,0) - i(u_0^+(x,0) - u_0^-(x,0))] = (1 + \varkappa_0)[\Phi^+(x) - \Phi^-(x)] \tag{18}$$

[μ is the shear modulus of the band material and $\varkappa_0 = 3 - 4\nu$; the zero superscript at the complex potential $\Phi^{(0)}(z)$ is omitted).

Using the Sokhotsky–Plemelj formulas [4] and taking into account Eq. (14), we find

$$\Phi^+(x) - \Phi^-(x) = \sqrt{(x - a_0)(x - b_0)} \Big(\frac{1}{\pi i} \int\limits_{a_0}^{b_0} \frac{f(t)\, dt}{\sqrt{(t - a_0)(t - b_0)}\,(t - x)} + 2P_n(x) \Big). \tag{19}$$

Substituting Eq. (19) into the right side of Eq. (18), taking into account Eq. (7), and applying some transformations, we obtain the complex nonlinear integrodifferential equation

$$-\frac{i(1 + \varkappa_0)}{2\mu} \sqrt{(b_0 - x)(x - a_0)} \Big(-\frac{1}{\pi} \int\limits_{a_0}^{b_0} \frac{f(t)\, dt}{\sqrt{(b_0 - t)(t - a_0)}\,(t - x)} + 2P_n(x) \Big)$$

$$= \frac{\partial}{\partial x} [C(x, \sigma^0)(q_y^{(0)}(x) - i q_{xy}^{(0)}(x))]. \tag{20}$$

In the case of pure bending (see Fig. 1a) and in the case of band bending by a uniformly distributed load (see Fig. 1b), we have $q_{xy}^{(0)}(x) = 0$ by virtue of load symmetry. In the case of cantilever beam bending (see Fig. 1c), normal $q_y^{(0)}$ and shear $q_{xy}^{(0)}$ forces arise in the bonds between the faces.

In the general case, separating the real and imaginary parts in Eq. (20), we obtain a system of nonlinear singular integrodifferential equations with respect to $q_y^{(0)}(x)$ and $q_{xy}^{(0)}(x)$ with a kernel of the Cauchy type kernel. These equations can be solved by using a collocation scheme with approximation of unknown functions. With the use of quadrature formulas of the Gauss formula type, the integrals in Eqs. (17) and (20) are replaced by finite sums, and the derivatives in the right side of Eq. (20) are replaced by finite-difference approximations. As a result, each integrodifferential equation is reduced to a system of algebraic equations with respect to the approximate values of $q_y^{(0)}(x)$ and $q_{xy}^{(0)}(x)$ at nodal points. As the size of the pre-fracture zone is unknown even in the particular case of linearly elastic bonds, the resultant algebraic systems are nonlinear. A method of consecutive approximations [5] is used to solved these systems of equations in the case of linear bonds. An iterative algorithm similar to the method of elastic solutions [6] is also used to determine the forces in the pre-fracture zone in the case of a nonlinear law of deformation of the bonds. The law of deformation of interparticle bonds (adhesion forces) is assumed to be linear: $V \leqslant V_*$. Subsequent iterations are performed if $V(x) > V_*$. To calculate such iterations, a system of equations for the bonds with the effective compliance varying along the pre-fracture zone and depending on the absolute value of the force vector in the bonds obtained at the previous step of calculations is solved in each approximation. The calculation of the effective compliance is similar to the calculation of the secant modulus in the method of variable parameters of elasticity [7]. The process of consecutive approximations is terminated when the forces in the pre-fracture zone obtained at two consecutive iterations differ insignificantly. In each approximation, the algebraic system was solved numerically by the Gauss method with the choice of the basic element.

Let us construct the problem solution in the first approximation. If there are volume forces, the solution of the plane problem is sought in the form

$$N_x^* = N_{x_*}^{(1)} + N_{x_1}^{(1)}, \qquad N_y^* = N_{y_*}^{(1)} + N_{y_1}^{(1)}, \qquad N_{xy}^* = N_{xy_*}^{(1)} + N_{xy_1}^{(1)},$$

where $N_{x_*}^{(1)}$, $N_{y_*}^{(1)}$, and $N_{xy_*}^{(1)}$ are the particular solution of the equations of the plane theory of elasticity in the presence of volume forces (4); $N_{x_1}^{(1)}$, $N_{y_1}^{(1)}$, and $N_{xy_1}^{(1)}$ are the general solution of the equations of the plane theory of elasticity in the absence of volume forces.

For the forces in the first approximation in the presence of volume forces, we have the following general presentations [8]:

$$\frac{N_x^* + N_y^*}{2h_0} = 4\,\mathrm{Re}\Big(\Phi^{(1)}(z) - \frac{1}{2(1+\varkappa_0)}\,\frac{\partial F_1}{\partial z}\Big),$$

$$\frac{N_y^* - N_x^* + 2iN_{xy}^*}{2h_0} = 2\Big(\bar{z}\Phi^{(1)\prime}(z) + \Psi^{(1)}(z) + \frac{1}{2(1+\varkappa_0)}\,\frac{\partial}{\partial z}\,(\varkappa_0 \bar{F}_1 - \bar{Q}_1)\Big).$$

These relations involve two analytical functions $\Phi^{(1)}(z)$ and $\Psi^{(1)}(z)$ of the complex variable $z = x + iy$ and two functions $F_1(z, \bar{z})$ and $Q_1(z, \bar{z})$, which are arbitrary particular solutions of the equations

$$\frac{\partial^2 F_1}{\partial z\,\partial \bar{z}} = F, \qquad \frac{\partial^2 Q_1}{\partial z^2} = \bar{F}, \tag{21}$$

where

$$F = X_1 + iY_1 = \frac{\partial \bar{h}}{\partial x}\,(N_x^{(0)} + iN_{xy}^{(0)}) + i\,\frac{\partial \bar{h}}{\partial y}\,(N_y^{(0)} - iN_{xy}^{(0)}).$$

To determine the complex potentials $\Phi^{(1)}(z)$ and $\Omega^{(1)}(z)$, we have the problem of linear coupling

$$\begin{gathered}[\Phi^{(1)}(x) + \Omega^{(1)}(x)]^+ + [\Phi^{(1)}(x) + \Omega^{(1)}(x)]^- = 2f_1(x), \\ [\Phi^{(1)}(x) - \Omega^{(1)}(x)]^+ - [\Phi^{(1)}(x) - \Omega^{(1)}(x)]^- = 0,\end{gathered} \tag{22}$$

where $a_1 \leqslant x \leqslant b_1$ is the affix of the points of the pre-fracture zone faces in the first approximation,

$$\begin{gathered}f_1(x) = f_0(t) - \bar{h}(q_y^{(0)} - iq_{xy}^{(0)}) + q_y^{(1)} - iq_{xy}^{(1)}, \\ f_0(x) = \frac{1}{1+\varkappa_0}\ \mathrm{Re}\,\frac{\partial F_1}{\partial z} - \frac{1}{2(1+\varkappa_0)}\Big(\varkappa_0\,\frac{\partial \bar{F}_1}{\partial z} - \frac{\partial \bar{Q}_1}{\partial z}\Big) \qquad \text{at} \quad y = 0.\end{gathered} \tag{23}$$

The general solution of the boundary-value problems has the form [4]

$$\Phi^{(1)}(z) = \Omega^{(1)}(z) = \frac{\sqrt{(z-a_1)(z-b_1)}}{2\pi i}\int_{a_1}^{b_1} \frac{f_1(t)\,dt}{\sqrt{(t-a_1)(t-b_1)}\,(t-z)}. \tag{24}$$

The parameters a_1 and b_1 determining the pre-fracture zone in the first approximation are found from the condition of solvability of the boundary-value problem (22):

$$\int_{a_1}^{b_1} \frac{f_1(t)\,dt}{\sqrt{(t-a_1)(t-b_1)}} = 0, \qquad \int_{a_1}^{b_1} \frac{t f_1(t)\,dt}{\sqrt{(t-a_1)(t-b_1)}} = 0. \tag{25}$$

Relations (24) and (25) include the unknown stresses $q_y^{(1)}(x)$ and $q_{xy}^{(1)}(x)$ in the bonds between the pre-fracture zone faces, which are found with the use of the additional condition (8).

Using solution (24), we find the derivative of the opening of the pre-fracture zone faces in the first approximation:

$$\frac{\partial}{\partial x}\,[v_1^+(x,0) - v_1^-(x,0) - i(u_1^+(x,0) - u_1^-(x,0))] = -\frac{1+\varkappa_0}{2\pi\mu}\,\sqrt{(b_1-x)(x-a_1)}\int_{a_1}^{b_1} \frac{f_1(t)\,dt}{\sqrt{(b_1-t)(t-a_1)}\,(t-x)}.$$

To determine the forces $q_y^{(1)} - iq_{xy}^{(1)}$ in the bonds in the first approximation, we obtain the following complex nonlinear integrodifferential equation:

$$-\frac{i(1+\varkappa_0)}{2\pi\mu}\,\sqrt{(b_1-x)(x-a_1)}\int_{a_1}^{b_1} \frac{f_1(t)\,dt}{\sqrt{(b_1-t)(t-a_1)}\,(t-x)} = \frac{\partial}{\partial x}\,[C(x,\sigma^1)(q_y^{(1)}(x) - iq_{xy}^{(1)}(x))].$$

To calculate the external load at which crack nucleation occurs, the problem formulation should be supplemented with a condition (criterion) of crack nucleation (breakdown of interparticle bonds of the material). As such a condition, we use the criterion of the critical opening of the pre-fracture zone faces

$$|v^+ - v^- - i(u^+ - u^-)| = \delta_{\rm cr},$$

where $\delta_{\rm cr}$ characterizes the band material resistance to cracking. This additional condition allows us to find the parameters of the bent band at which crack nucleation occurs.

Numerical Solution and Analysis of Results. As in the zeroth approximation, applying some transformations, we obtain a system of nonlinear integrodifferential equations with respect to the unknown functions $q_y^{(1)}$ and $q_{xy}^{(1)}$:

$$-\frac{1+\varkappa_0}{2\pi\mu}\,\sqrt{(b_1-x)(x-a_1)}\int_{a_1}^{b_1} \frac{f_1^*(t)\,dt}{\sqrt{(b_1-t)(t-a_1)}\,(t-x)} = \frac{\partial}{\partial x}\,[C(x,\sigma^1)q_y^{(1)}(x)]; \tag{26}$$

$$-\frac{1+\varkappa_0}{2\pi\mu}\,\sqrt{(b_1-x)(x-a_1)}\int_{a_1}^{b_1} \frac{f_1^{**}(t)\,dt}{\sqrt{(b_1-t)(t-a_1)}\,(t-x)} = \frac{\partial}{\partial x}\,[C(x,\sigma^1)q_{xy}^{(1)}(x)] \tag{27}$$

[$f_1^*(t) = \operatorname{Re} f_1(t)$ and $f_1^{**}(t) = \operatorname{Im} f_1(t)$].

As in the zeroth approximation, each of the equations (26) and (27) is a nonlinear integrodifferential equation with a kernel of the Cauchy type kernel and can be solved only numerically.

To determine the stress distribution and the size of the pre-fracture zone, we have to define the law of variation of the band thickness. We expand the function $h(x,y)$ into a Taylor series in the origin and keep two terms in this expansion:

$$h(x,y) = h_0[1 + \varepsilon(a_* x + b_* y)] \tag{28}$$

($2h_0$ is the band thickness in the origin; a_* and b_* are certain coefficients). Then, the expressions for the volume force components in the first approximation acquire the form

$$X_1 = a_* N_x^{(0)} + b_* N_{xy}^{(0)}, \qquad Y_1 = a_* N_{xy}^{(0)} + b_* N_y^{(0)}.$$

Using the formulas

$$\frac{N_x^{(0)}+N_y^{(0)}}{2h_0}=4\,\mathrm{Re}\,\Phi^{(0)}(z),\qquad \frac{N_y^{(0)}-iN_{xy}^{(0)}}{2h_0}=\Phi^{(0)}(z)+\Omega^{(0)}(\bar z)+(z-\bar z)\,\overline{\Phi^{(0)\prime}(z)},$$

we determine the force components $N_x^{(0)}$, $N_y^{(0)}$, and $N_{xy}^{(0)}$. Then, using Eqs. (4), we find the function $F=X_1+iY_1$.

Integrating Eqs. (21), we obtain

$$F_1(z,\bar z)=\int^{z}dz\int^{\bar z}F(z,\bar z)\,d\bar z,\qquad Q_1(z,\bar z)=\int^{z}dz\int^{z}\overline{F(z,\bar z)}\,dz.$$

Using the found functions $F_1(z,\bar z)$ and $Q_1(z,\bar z)$, we determine the function $f_0(x)$ from Eq. (23). Then, using formulas (24) and (25), we find the solution of the boundary-value problem in the first approximation.

The integral equations (26) and (27) with the additional conditions (25) are replaced by algebraic equations. First, all intervals of integration in the integral equations (26) and (27) and in the additional conditions (25) are reduced to one interval $[-1,1]$ by replacing the variables

$$t=\frac{a_1+b_1}{2}+\frac{b_1-a_1}{2}\,\tau,\qquad x=\frac{a_1+b_1}{2}+\frac{b_1-a_1}{2}\,\eta.$$

With such replacement of variables, the left side of the integrodifferential equation (26) acquires the form

$$-\frac{1}{\pi}\sqrt{1-\eta^2}\,\Big(\int_{-1}^{1}\frac{q_y^{(1)}(\tau)\,d\tau}{\sqrt{1-\tau^2}\,(\tau-\eta)}+\int_{-1}^{1}\frac{f_y(\tau)\,d\tau}{\sqrt{1-\tau^2}\,(\tau-\eta)}\Big);\tag{29}$$

correspondingly, for the left side of Eq. (27), we obtain

$$-\frac{1}{\pi}\sqrt{1-\eta^2}\,\Big(\int_{-1}^{1}\frac{q_{xy}^{(1)}(\tau)\,d\tau}{\sqrt{1-\tau^2}\,(\tau-\eta)}+\int_{-1}^{1}\frac{f_{xy}(\tau)\,d\tau}{\sqrt{1-\tau^2}\,(\tau-\eta)}\Big).\tag{30}$$

Here, $f_y=f_1^*-q_y^{(1)}$ and $f_{xy}=f_1^{**}-q_{xy}^{(1)}$.

The derivatives involved into the right sides of Eqs. (26) and (27) in an arbitrary internal node are replaced by the finite-difference approximation [9]. Thereby, we take into account the boundary conditions $q_y^{(1)}(a_1)=q_y^{(1)}(b_1)=0$ and $q_{xy}^{(1)}(a_1)=q_{xy}^{(1)}(b_1)=0$ at $\eta_0=\pm1$ [which corresponds to the conditions $v_1^+(a_1,0)-v_1^-(a_1,0)=0$, $v_1^+(b_1,0)-v_1^-(b_1,0)=0$, $u_1^+(a_1,0)-u_1^-(a_1,0)=0$, and $u_1^+(b_1,0)-u_1^-(b_1,0)=0$]. Using the quadrature formula

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{g(\tau)\,d\tau}{\sqrt{1-\tau^2}\,(\tau-\eta)}=\frac{1}{n\sin\theta}\sum_{k=1}^{n}g_k\sum_{m=0}^{n-1}\cos\theta_k\sin m\theta,$$

where $\tau=\cos\theta$, $\eta_m=\cos\theta_m$, and $\theta_m=\pi(2m-1)/(2n)$ $(m=1,2,\ldots,n)$, we replace the integrals in Eqs. (29) and (30) by finite sums, and the derivatives in the right sides of Eqs. (26) and (27) are replaced by finite-difference approximations. With the formulas given above, each integrodifferential equation can be replaced by a system of algebraic equations with respect to the approximate values of the sought function at nodal points. The corresponding algebraic systems have the form

$$\sum_{\nu=1}^{n}A_{m\nu}(q_{y,\nu}^{(1)}+f_{y,\nu})=\frac{(1+\varkappa_0)n}{4\mu(b_1-a_1)}\,[C(x_{m+1},\sigma^1)q_{y,m+1}^{(1)}-C(x_{m-1},\sigma^1)q_{y,m+1}^{(1)}]\qquad(m=1,2,\ldots,n),$$

$$\sum_{\nu=1}^{n}A_{m\nu}(q_{xy,\nu}^{(1)}+f_{xy,\nu})=\frac{(1+\varkappa_0)n}{4\mu(b_1-a_1)}\,[C(x_{m+1},\sigma^1)q_{xy,m+1}^{(1)}-C(x_{m-1},\sigma^1)q_{xy,m+1}^{(1)}]\qquad(m=1,2,\ldots,n),$$

where

$$q_{y,\nu}^{(1)}=q_y^{(1)}(\tau_\nu),\qquad q_{xy,\nu}^{(1)}=q_{xy}^{(1)}(\tau_\nu),\qquad f_{y,\nu}=f_y(\tau_\nu),\qquad f_{xy,\nu}=f_{xy}(\tau_\nu),$$

$$x_{m+1}=\frac{a_1+b_1}{2}+\frac{b_1-a_1}{2}\,\eta_{m+1},\qquad A_{m\nu}=-\frac{1}{n}\,\cot\,\frac{\theta_m\mp\theta_\nu}{2}$$

(in the last expression, the upper (lower) sign refers to the case of an odd (even) value of $|m-\nu|$).

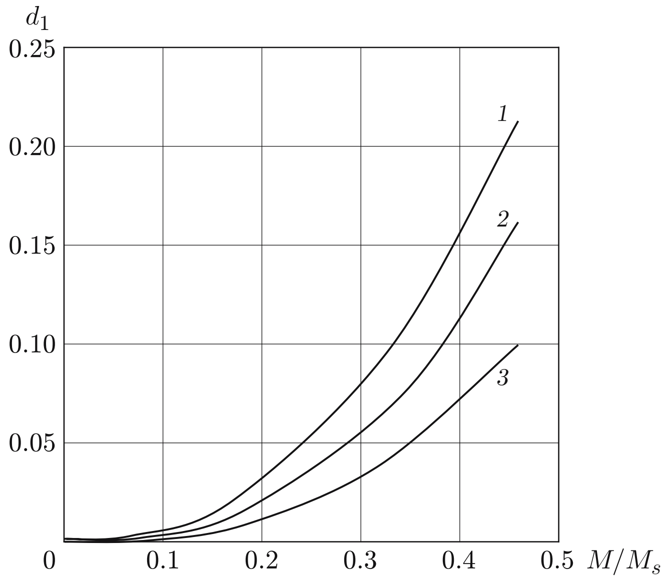

Fig. 2

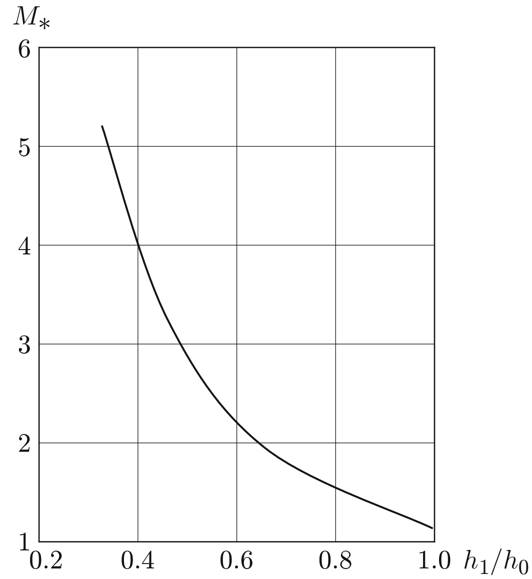

Fig. 3

Fig. 2. Pre-fracture zone length d_1 versus the dimensionless load M/M_s in the case of pure bending of the band: $h_1/h_0 = 0.5$ (1), 0.6 (2), and 0.75 (3).

Fig. 3. Dimensionless critical load M_* versus the dimensionless band thickness h_1/h_0 in the case of pure bending.

Let us perform algebraization of the solvability conditions of the boundary-value problem (25). Separating the real and imaginary parts in these conditions, replacing the variables, and using the Gauss quadrature formula, we obtain the problem solvability conditions in the form

$$\frac{\pi}{n}\sum_{\nu=1}^{n} f_y(\cos\theta_\nu) = 0, \qquad \sum_{\nu=1}^{n} \tau_\nu f_y(\tau_\nu) = 0,$$

$$\frac{\pi}{n}\sum_{\nu=1}^{n} f_{xy}(\cos\theta_\nu) = 0, \qquad \sum_{\nu=1}^{n} \tau_\nu f_{xy}(\tau_\nu) = 0.$$

As a result of algebraization, instead of each integral equation with the corresponding boundary conditions, we obtain $n+2$ algebraic equations for determining the stresses at nodal points and the pre-fracture zone size. The method used to solve such systems is described above.

With allowance for Eq. (2), the critical condition is written in the following form (for $x = x_0$):

$$C(x_0, \sigma(x_0))\sigma(x_0) = \delta_{\rm cr}. \tag{31}$$

The joint solution of the resultant equations and condition (31) with specified characteristics of the bonds allows us to determine the critical external load and the pre-fracture zone size $l_{\rm cr}$ in the critical equilibrium state at which a crack nucleates.

Figure 2 shows the pre-fracture zone length $d_1 = (b-a)/(2c)$ as a function of the dimensionless load M/M_s ($M_s = \sigma_s h_0^2/4$, where σ_s is the yield stress of the band material under tension) in the case of pure bending of the beam whose thickness follows law (28). The following values of the parameters were used in the calculations: $\nu = 0.3$ and $n = 30$. Figure 3 shows the dimensionless critical load

$$M_* = \frac{3}{\sqrt{2}}\,\frac{M}{h_0 c^{3/2}}\,\frac{1}{\sqrt{E\sigma_s\delta_{\rm cr}}}$$

as a function of the dimensionless band thickness h_1/h_0.

An analysis of the critical equilibrium state of the variable-thickness band in which crack nucleation occurs reduces to a parametric study of the resultant algebraic systems and the criterion of crack nucleation for different laws of deformation of the bonds, elastic constants of the material, and geometric characteristics of the band. The forces in the bonds and the opening of the pre-fracture zone faces are determined in each approximation directly from the solution of the algebraic systems.

Thus, an effective algorithm of solving problems of fracture mechanics on crack nucleation in a variable-thickness band under external forcing was developed, which allows the solution in each approximation to be constructed in a unified manner by the perturbation method.

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